Journal of Statistical Physics, Vol. 35, Nos. 3/4, 1984

On the Invariance of Constitutive Equations According to the Kinetic Theory of Gases

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Received October 11, 1983; revision received December 8, 1983

Iterative techniques for solving the Boltzmann equation in the kinetic theory of gases yield expressions for the stress tensor and heat flux vector that are analogous to constitutive equations in continuum mechanics. However, these expressions are not generally invariant under the Euclidean group of transformations, whereas constitutive equations in continuum mechanics are usually required to be by the principle of material frame indifference. This disparity in invariance properties has led some previous investigators to argue that Euclidean invariance should be discarded as a contraint on constitutive equations. It is proven mathematically in this paper that the results of the Chapman–Enskog iterative procedure have no direct bearing on this issue. In order to settle this question, it is necessary to examine mathematically the effect of superimposed rigid body rotations on solutions of the Boltzmann equation. A preliminary investigation along these lines is presented which suggests that the kinetic theory is consistent with material frame indifference in at least a strong approximate sense provided that the disparity in the time scales of the microscopic and macroscopic motions is extremely large—a condition which is usually a prerequisite for the existence of constitutive equations.

KEY WORDS: Kinetic theory; continuum mechanics; Chapman–Enskog method; Burnett equations; invariance properties; and material frame indifference.

1. INTRODUCTION

The Chapman–Enskog iterative procedure in the kinetic theory of gases, which has its origin around the turn of the century, represented the first

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0022-4715/84/0500-0457$03.50/0

useful method for solving the Boltzmann equation by successive approximations. In the first approximation the Euler equation is obtained, in the second approximation the Navier–Stokes equations are obtained, and in the third approximation the Burnett equations are obtained (cf., Chapman and Cowling[1]). Several decades later, alternative approaches such as the 13-moment expansion of Grad (see Grad[2]) and the Maxwellian iteration (see Ikenberry and Truesdell[3]) have been developed. All of these approaches yield expressions for the stress tensor and heat flux vector that are analogous to constitutive equations in continuum mechanics. More specifically, algebraic equations are obtained that tie the stress tensor and heat flux vector to the velocity and temperature fields as well as their time derivatives and spatial gradients. However, these expressions are not generally invariant under the Euclidean group of transformations (i.e., under arbitrary time-dependent rotations and translations of the spatial frame of reference), whereas constitutive equations in continuum mechanics are usually required to be by the principle of material frame indifference.[4] This disparity in invariance properties has led numerous investigators during the past decade to argue that the principle of material frame indifference should be discarded (see Müller,[5] Edelen and McLennan,[6] Söderholm,[7] and Woods[8]). It should be noted at this point that the concept of material frame indifference (i.e., the Euclidean invariance of constitutive equations), which has its roots in the nineteenth century starting with Boussinesq, is tied to the physical idea that stress is associated exclusively with deformations and, hence, does not depend on the motion of the observer. Of course, the theory of elasticity, Navier–Stokes theory, and several other classical continuum theories satisfy this principle identically. Furthermore, since the principle of material frame indifference places severe restrictions on the allowable form of constitutive equations, it has served as a useful tool in the development of modern continuum theories.

The purpose of the present paper is to examine this question of invariance in the kinetic theory in more detail. A preliminary investigation along these lines was presented in Speziale.[9] Here, it will be proven mathematically that the assumption in the Chapman–Enskog iterative procedure that time does not enter as an explicit argument in the expression for the local time rate of change of the macroscopic velocity is not closed with respect to Euclidean transformations. To be specific, if a given macroscopic process is a member of the Chapman–Enskog class, one which differs by an arbitrary time-dependent rotation will not be. This gives rise to the lack of Euclidean invariance in the Burnett approximation. However, as a result of the lack of closure and the fact that this assumption (which is part of what is usually termed as the hydrodynamic description) is not a rigorous consequence of the Boltzmann equation or the general mathematical structure of the kinetic theory, it is simply unrigorous to draw any direct physical conclusions about the resulting lack of Euclidean invariance. In order to resolve this question concerning the consistency of material frame indifference with the kinetic theory it is necessary to examine the physical consequences of this principle within the framework of continuum mechanics and compare with corresponding physical results that are derived directly from the Boltzmann equation. It will be demonstrated that in order to accomplish this task it is necessary to examine the effect of superimposed rigid body motions of the gas on solutions of the Boltzmann equation. This is necessary since the main physical consequence of material frame indifference within the framework of continuum mechanics is that it forbids the *values* of the stress tensor and heat flux vector to be altered by a superimposed rigid body motion of the material. It will be shown that provided the disparity in the time scales of the microscopic and macroscopic motions is extremely large (as is usually the case in continuum mechanics) the kinetic theory of gases is consistent with material frame indifference in at least a strong approximate sense. Furthermore, it will also be shown that while there is cause to question the general consistency of material frame indifference with the kinetic theory (this doubt arises for different reasons than those presented by the authors mentioned above), nothing final can be proven at this time because of the lack of exact solutions to the Boltzmann equation.

2. EUCLIDEAN INVARIANCE IN CONTINUUM MECHANICS AND THE KINETIC THEORY

In continuum mechanics, the governing field equations are those for the conservation of mass, linear momentum, angular momentum, and energy which are given by (cf., Truesdell and Noll[4])

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_k) = 0 \tag{1}$$

$$\rho \dot{v}_k = \frac{\partial T_{kl}}{\partial x_l} + \rho b_k \tag{2}$$

$$T_{kl} = T_{lk} \tag{3}$$

$$\rho \dot{\epsilon} = T_{kl} \frac{\partial v_l}{\partial x_k} - \frac{\partial q_k}{\partial x_k} + \rho h \tag{4}$$

where ρ is the mass density, $\mathbf{v}$ is the velocity field, $\mathbf{T}$ is the stress tensor, $\mathbf{b}$ is the external body force per unit mass, ϵ is the internal energy per unit mass, $\mathbf{q}$ is the heat flux vector, and h is the energy supply per unit mass. In (2) and (4), a superposed dot denotes the material time derivative, and the Einstein summation convention applies to repeated indices. Furthermore, by the symmetry of the stress tensor in (3) we have restricted our attention to nonpolar continua for which there are no body couples or couple stresses. The system of equations (1)–(4) are not closed and must be supplemented with constitutive equations for $\mathbf{T}$, $\mathbf{q}$, and ϵ which represent a mathematical description of the material properties. These constitutive equations for thermomechanical materials require that the stress, heat flux, and internal energy at any material point be functionals in the history of the motion and temperature of all points of the material.[4] For many important applications, these general constitutive equations reduce to the algebraic form mentioned earlier. Constitutive equations are usually subject to the restrictions that arise from the second law of thermodynamics and from material frame indifference. The latter constraint requires that they be form invariant under the Euclidean group of transformations, i.e., constitutive equations for $\mathbf{T}$, $\mathbf{q}$, and ϵ must tranform as

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T, \qquad \mathbf{q}^* = \mathbf{Q}\mathbf{q}, \qquad \epsilon^* = \epsilon \tag{5}$$

for any two motions which differ by the Euclidean group of transformations

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{d}(t), \qquad t^* = t + a \tag{6}$$

where a is an arbitrary constant, $\mathbf{d}(t)$ is an arbitrary time-dependent vector, and $\mathbf{Q}(t)$ is an arbitrary time-dependent proper orthogonal tensor so that

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}, \qquad |\mathbf{Q}| = 1 \tag{7}$$

In (5) and (7), the superscript T denotes the transpose, whereas in (7) $\mathbf{I}$ denotes the unit tensor and $|\cdot|$ denotes the determinant. The Euclidean group of transformations, which consist of arbitrary time-dependent rotations and translations of the spatial frame of reference and shifts in the origin of time, represent the most general change of frame (or observer) in Euclidean space.

The continuum theory described above is complete. By completeness, it is meant that the domain of the stress, heat flux, and internal energy are formed of all possible motion and temperature histories (see Wang[10]). As a consequence of (5) and completeness, the violation of material frame indifference leads to a situation where the *values* of the stress tensor and the heat flux vector can be altered by a rigid body motion of the material.[4] Within the axiomatic framework of continuum mechanics, the main physical consequence of material frame indifference is that it forbids such an occurrence.

The kinetic theory of gases is based on a statistical description controlled by the phase density function $f(\mathbf{x}, \mathbf{c}, t)$ where the position vector $\mathbf{x}$ and molecular velocity $\mathbf{c}$ forms a six-dimensional space referred to as the phase space. From the phase density function, various macroscopic fields are definable, e.g., the number density and macroscopic velocity field are given by

$$n(\mathbf{x}, t) = \int_{S_c} f(\mathbf{x}, \mathbf{c}, t)\, d^3c \tag{8}$$

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{n} \int_{S_c} \mathbf{c} f(\mathbf{x}, \mathbf{c}, t)\, d^3c \tag{9}$$

where S_c is the space of all molecular velocities. The fluctuating velocity $\mathbf{u}$, also referred to as the peculiar velocity, is defined as follows:

$$\mathbf{c} = \mathbf{v} + \mathbf{u} \tag{10}$$

Making use of the fluctuating velocity the stress tensor, heat flux vector, and internal energy density are, respectively, defined by

$$T_{kl} = -m \int_{S_c} u_k u_l f\, d^3c \tag{11}$$

$$q_k = \frac{m}{2} \int_{S_c} \mathbf{u} \cdot \mathbf{u} u_k f\, d^3c \tag{12}$$

$$\epsilon = \frac{1}{2n} \int_{S_c} \mathbf{u} \cdot \mathbf{u} f\, d^3c \tag{13}$$

where m is the molecular mass (the macroscopic mass density $\rho = mn$). The absolute temperature θ is given by

$$\theta = \frac{m}{3kn} \int_{S_c} \mathbf{u} \cdot \mathbf{u} f\, d^3c \tag{14}$$

where k is the Boltzmann constant. Here, it should be noted that as a direct consequence of (11)–(14), $\mathbf{T}$, $\mathbf{q}$, ϵ, and θ are frame-indifferent tensors.(9) Furthermore, in the kinetic theory the internal energy density is identically given by

$$\epsilon = \frac{3}{2} \frac{k}{m} \theta \tag{15}$$

For a slightly rarefied monatomic gas, the phase density function is determined from the Boltzmann equation which takes the form

$$\frac{\partial f}{\partial t} + c_k \frac{\partial f}{\partial x_k} + b_k \frac{\partial f}{\partial c_k} = \int_{S_{\hat{c}}} \int_{S_\Omega} (f' \hat{f}' - f \hat{f}) |\hat{\mathbf{c}} - \mathbf{c}|\, d^2\Omega\, d^3\hat{c} \tag{16}$$

where f, $\hat{f}$, f', and $\hat{f}'$ are, respectively, the values of the phase density corresponding to the arguments $\mathbf{c}$, $\hat{\mathbf{c}}$, $\mathbf{c}'$, and $\hat{\mathbf{c}}'$, and $d^2\Omega$ is the differential cross section of molecular encounters which depends on the molecular model considered. Here, we are considering only binary collisions where a molecule with velocity $\mathbf{c}$ and a molecule with velocity $\hat{\mathbf{c}}$ collide and then assume the new velocities $\mathbf{c}'$ and $\hat{\mathbf{c}}'$, respectively, which are related by the laws of conservation of linear momentum and energy.[1] Balance laws for mass, linear momentum, and energy are obtained by taking moments of the Boltzmann equation with m, $m\mathbf{c}$, and $m\mathbf{c}\cdot\mathbf{c}$, respectively.[1] These equations are identical to the balance laws of continuum mechanics with one exception—the energy equation has no energy supply h. As a result of the absence of an energy supply, the kinetic theory is not complete, i.e., the domain of the stress tensor, heat flux vector, and internal energy density are *not* formed of all possible motion and temperature histories. There are some motion and temperature histories that are incompatible with (4) when $h = 0$ and hence do not correspond to any solution of the Boltzmann equation.[10]

Various iterative procedures for solving the Boltzmann equation such as the Chapman–Enskog method and the Maxwellian iteration yield expressions for the stress and heat flux that are analogous to constitutive equations in continuum mechanics. In the first approximation, these approaches yield the Euler equation with a zero heat flux, while in the second approximation, Navier–Stokes theory and the Fourier law for heat conduction are obtained. The first and second approximations are form invariant under the Euclidean group of transformations and, hence, satisfy the principle of material frame indifference. The third approximation for the Chapman–Enskog method yields the Burnett equations. These equations are *not* invariant under the Euclidean group of transformations.[5–8] The significance of this lack of Euclidean invariance will now be examined.

The Chapman–Enskog iterative procedure is based on the assumption that the phase density function f and the partial derivatives with respect to time of the number density n, absolute temperature θ, and macroscopic velocity $\mathbf{v}$ are implicit functions of time only through n, θ, $\mathbf{v}$, and their spatial gradients, i.e.,

$$f = \Phi_1(\mathbf{x}, \mathbf{c}, n, \nabla n, \ldots, \theta, \nabla\theta, \ldots, \mathbf{v}, \nabla\mathbf{v}, \ldots) \tag{17}$$

$$\frac{\partial n}{\partial t} = \Phi_2(\mathbf{x}, n, \nabla n, \ldots, \theta, \nabla\theta, \ldots, \mathbf{v}, \nabla\mathbf{v}, \ldots) \tag{18}$$

$$\frac{\partial \theta}{\partial t} = \Phi_3(\mathbf{x}, n, \nabla n, \ldots, \theta, \nabla\theta, \ldots, \mathbf{v}, \nabla\mathbf{v}, \ldots) \tag{19}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{\Phi}_4(\mathbf{x}, n, \nabla n, \ldots, \theta, \nabla\theta, \ldots, \mathbf{v}, \nabla\mathbf{v}, \ldots) \tag{20}$$

Equations (17)–(20) constitute what is commonly referred to as the hydrodynamic description. In Speziale,[9] it was demonstrated that there exist a wide variety of functions Φ_1, Φ_2, and Φ_3 for which equations (17)–(19) are form invariant under the Euclidean group of transformations (6). However, no such function $\mathbf{\Phi}_4$ exists which is, in general, Euclidean invariant. This can be seen by a direct analysis of the Euclidean transformation of (20) which is of the form[9]

$$\begin{aligned}\frac{\partial v_k^*}{\partial t^*} &= Q_{kl}\Phi_{4l}(\mathbf{x}^*, n^*, \nabla^* n^*, \ldots, \theta^*, \nabla^*\theta^*, \ldots, \mathbf{v}^*, \nabla^*\mathbf{v}^*, \ldots \mathbf{Q}, \dot{\mathbf{Q}}, \mathbf{d}, \dot{\mathbf{d}}) \\ &\quad + \dot{Q}_{kl}\left[Q_{ml}v_m^* + \dot{Q}_{ml}(x_m^* - d_m) - Q_{ml}\dot{d}_m\right] \\ &\quad + \ddot{Q}_{kl}Q_{ml}(x_m^* - d_m) + \ddot{d}_k - \frac{\partial v_k^*}{\partial x_l^*}\left[\dot{Q}_{lm}Q_{nm}(x_n^* - d_n) + \dot{d}_l\right]\end{aligned} \tag{21}$$

At a particular instant of time, $\ddot{\mathbf{Q}}$ and $\ddot{\mathbf{d}}$ can be varied independently of $\dot{\mathbf{Q}}$, $\mathbf{Q}$, $\dot{\mathbf{d}}$, and $\mathbf{d}$ and, consequently, (21) will be an *explicit* function of time in contradiction of the Chapman–Enskog hydrodynamic assumption. For example, we can take

$$\begin{aligned}\mathbf{Q}(t) &= \begin{bmatrix} \cos\psi(t) & \sin\psi(t) & 0 \\ -\sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \psi(t) &= t(t-t_0)^2\alpha, \qquad \mathbf{d}(t) = t(t-t_0)^2\boldsymbol{\beta}\end{aligned} \tag{22}$$

where $\boldsymbol{\beta}$ is any constant vector and α is any constant. Then, at time t_0 we have

$$\mathbf{Q}(t_0) = \mathbf{I}, \qquad \dot{\mathbf{Q}}(t_0) = \mathbf{0} \tag{23}$$

$$\ddot{\mathbf{Q}}(t_0) = \begin{bmatrix} 0 & 2\alpha t_0 & 0 \\ -2\alpha t_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{24}$$

$$\mathbf{d}(t_0) = \mathbf{0}, \qquad \dot{\mathbf{d}}(t_0) = \mathbf{0}, \qquad \ddot{\mathbf{d}}(t_0) = 2t_0\boldsymbol{\beta} \tag{25}$$

where $\mathbf{I}$ is the unit tensor. The substitution of (23)–(25) into (21) yields

$$\begin{aligned}\frac{\partial \mathbf{v}^*}{\partial t^*}\Big|_{t=t_0} &= \mathbf{\Phi}_4(\mathbf{x}^*, n^*, \nabla^* n^*, \ldots, \theta^*, \nabla^*\theta^*, \ldots, \mathbf{v}^*, \nabla^*\mathbf{v}^*, \ldots,)\Big|_{t=t_0} \\ &\quad - 2t_0\boldsymbol{\alpha} \times \mathbf{x}^* + 2t_0\boldsymbol{\beta}\end{aligned} \tag{26}$$

where $\boldsymbol{\alpha} = \alpha\mathbf{e}_3^*$ and $\mathbf{e}_3^*$ is a unit vector in the x_3^* direction. Hence, no matter what choice is made for the function $\mathbf{\Phi}_4$, Eq. (26) will be an *explicit* function of time in violation of the Chapman–Enskog hydrodynamic assumption. It is thus clear that the Chapman–Enskog iterative technique is not closed with respect to Euclidean transformations (i.e., arbitrary time-dependent superimposed rigid body motions). To be more specific, if a given macroscopic process is a member of the Chapman–Enskog class, then one which differs by an arbitrary time-dependent superimposed rigid body motion will not be. Since a general Euclidean transformation takes us outside of the Chapman–Enskog class, the lack of Euclidean invariance of results obtained from this technique is of no physical consequence in and of itself. This can be illustrated by the following simple example: the Bernoulli equation

$$p + \tfrac{1}{2}\rho v^2 = \text{const} \tag{27}$$

for the steady irrotational flow of an inviscid and incompressible fluid in the absence of body forces. Equation (27) is *not* form invariant under a Galilean transformation. However, this is of no physical consequence since (27) is not closed with respect to the Galilean transformations (i.e., the Galilean transformation of a steady flow is *not* in general steady) and, hence, a Galilean transformation takes us outside of its domain of applicability. The use of the lack of Euclidean invariance of results obtained from the Chapman–Enskog iterative procedure as the sole argument against material frame indifference is thus tantamount to the following fallacious argument: the Bernoulli equation (27), an equation with both empirical and theoretical confirmation, is *not* form invariant under the Galilean group of transformations; thus Galilean invariance cannot be a general principle of classical physics.

As alluded to earlier, the main physical consequence of material frame indifference is that it forbids the *values* of the stress tensor and heat flux vector to be altered by a rigid body motion of the material. Here, unlike in continuum mechanics, there can be a difference between a change of observer (for which the stress and heat flux are invariant) and a superimposed rigid body motion as a result of the way in which the body force appears in the Boltzmann equation. Thus, to begin to address this question concerning the consistency of the kinetic theory with material frame indifference, the effect of rigid body motions on solutions of the Boltzmann equation must be determined. This can be accomplished by utilizing the Euclidean transformation of the Boltzman equation which takes the form (see Wang[11])

$$\frac{\partial f^*}{\partial t^*} + c_k^* \frac{\partial f^*}{\partial x_k^*} + \bar{b}_k^* \frac{\partial f^*}{\partial c_k^*} = \int_{S_{\hat{c}^*}}\int_{S_{\Omega^*}} (f^{*\prime}\hat{f}^{*\prime} - f^*\hat{f}^*)|\hat{\mathbf{c}}^* - \mathbf{c}^*|\, d^2\Omega^*\, d^3\hat{c}^* \tag{28}$$

where

$$\begin{aligned}\bar{b}_k^* = b_k^* + 2\dot{Q}_{kl}Q_{ml}c_m^* + 2\dot{Q}_{kl}\dot{Q}_{ml}(x_m^* - d_m)\\ - 2\dot{Q}_{kl}Q_{ml}\dot{d}_m + \ddot{Q}_{kl}Q_{ml}(x_m^* - d_m) + \ddot{d}_k\end{aligned} \tag{29}$$

and

$$b_k^* = Q_{kl} b_l \tag{30}$$

since forces are frame independent objects. Hence, it is clear that the Boltzmann equation is *not* form invariant under the Euclidean group of transformations. Its invariance group is no more than the Galilean group of transformations

$$\mathbf{x}^* = \mathbf{R}\mathbf{x} + \mathbf{V}t + \mathbf{C} \tag{31}$$

(where **R** is any constant proper orthogonal tensor and **V** and **C** are constant vectors) since under (31) we have

$$\ddot{\mathbf{Q}} = \dot{\mathbf{Q}} = \mathbf{0}, \qquad \ddot{\mathbf{d}} = 0 \tag{32}$$

and, thus, the apparent forces on the right-hand side of (29) vanish leaving $\bar{\mathbf{b}}^* = \mathbf{b}^*$. Despite the fact that the invariance group of the Boltzmann equation is no more than the Galilean group, it would be a mistake to conclude that the same is true of constitutive equations. Constitutive equations represent, at best, special solutions to the Boltzmann equation. It is a well established fact that special solutions can have a different invariance group.

In order to simplify matters we will first consider the case of changes of frame that differ by an arbitrary time-dependent translation. For this case,

$$\ddot{\mathbf{Q}} = \dot{\mathbf{Q}} = \mathbf{0} \tag{33}$$

and, thus, the Boltzmann equation (28) takes the form

$$\frac{\partial f^*}{\partial t^*} + c_k^* \frac{\partial f^*}{\partial x_k^*} + \left(b_k^* + \ddot{d}_k\right) \frac{\partial f^*}{\partial c_k^*} = \mathbb{C}[f^*] \tag{34}$$

where $\mathbb{C}$ denotes the collision operator. Hence, if an additional body force $\mathbf{b}_A^*$ is applied in (34) so that

$$\mathbf{b}_A^* = -\ddot{\mathbf{d}} \tag{35}$$

then it is clear that $f^* = f$ and therefore

$$n^* = n, \qquad \theta^* = \theta \tag{36}$$

$$\rho^* = \rho, \qquad \mathbf{c}^* = \mathbf{c} + \dot{\mathbf{d}}, \qquad \mathbf{v}^* = \mathbf{v} + \dot{\mathbf{d}} \tag{37}$$

$$T_{kl}^* = T_{kl}, \qquad q_k^* = q_k, \qquad \epsilon^* = \epsilon \tag{38}$$

Equations (35)–(38) have the following physical interpretation: If an arbitrary time-dependent rigid body translation is superimposed on a given macroscopic motion and temperature history, the *values* of the stress tensor, heat flux vector, and internal energy density are left unaffected. Furthermore, this rigid body translation is achieved by constructing the same process in a translating frame of reference in which an additional body force of the amount $-\ddot{\mathbf{d}}$ is applied. These results are completely identical with those obtained from continuum mechanics where the principle of material frame indifference is invoked. The kinetic theory of gases is therefore consistent with material frame indifference in so far as time-dependent translational accelerations of the spatial frame of reference are concerned. Consequently, the kinetic theory suggests that the invariance group for constitutive equations should be at least as much as the extended Galilean group of transformations

$$\mathbf{x}^* = \mathbf{R}\mathbf{x} + \mathbf{d}(t) \tag{39}$$

where $\mathbf{R}$ is any constant proper orthogonal tensor. Interestingly enough, results obtained from the Chapman–Enskog iterative procedure do exhibit invariance under (39). This results from the fact that the term containing $\ddot{\mathbf{d}}$ in (21) is spatially homogeneous.[9]

Now, we will consider the case of pure time-dependent rotations of the spatial frame of reference for which there is a problem. When material frame indifference is invoked, the field equations of continuum mechanics (1), (2), and (4) take the noninertial form

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial(\rho^* v_k^*)}{\partial x_k^*} = 0 \tag{40}$$

$$\rho^* \dot{v}_k^* = \frac{\partial T_{kl}^*}{\partial x_l^*} + \rho^*\left[b_k^* + 2\dot{Q}_{kl}Q_{ml}v_m^* + \left(2\dot{Q}_{kl}\dot{Q}_{ml} + \ddot{Q}_{kl}Q_{ml}\right)x_m^*\right] \tag{41}$$

$$\rho^* \dot{\epsilon}^* = T_{kl}^* \frac{\partial v_l^*}{\partial x_k^*} - \frac{\partial q_k^*}{\partial x_k^*} + \rho^* h^* \tag{42}$$

where

$$x_k^* = Q_{kl}(t)x_l \tag{43}$$

$$b_k^* = Q_{kl}(t)b_l \tag{44}$$

Equations (40)–(42) imply that a superimposed rigid body rotation can be obtained by constructing the same process in a rotating frame of reference where an additional body force $\mathbf{b}_A^*$ is applied in (41) which is given by

$$(\mathbf{b}_A^*)_k = -2\dot{Q}_{kl}Q_{ml}v_m^* - \left(2\dot{Q}_{kl}\dot{Q}_{ml} + \ddot{Q}_{kl}Q_{ml}\right)x_m^* \tag{45}$$

Then, of course, as a result of material frame indifference the values of the stress tensor, heat flux vector, and internal energy density will be unaffected. However, the Boltzmann equation of the kinetic theory appears to be inconsistent with this result. To be specific, the Boltzmann equation in a rotating frame of reference in which an additional body force given by (45) is applied takes the form

$$\frac{\partial f^*}{\partial t^*} + c_k^* \frac{\partial f^*}{\partial x_k^*} + b_k^* \frac{\partial f^*}{\partial c_k^*} + 2\dot{Q}_{kl} Q_{ml} (c_m^* - v_m^*) \frac{\partial f^*}{\partial c_k^*}$$
$$= \int_{S_{\hat{c}^*}} \int_{S_{\Omega^*}} (f^{*\prime} \hat{f}^{*\prime} - f^* \hat{f}^*) |\hat{c}^* - c^*| \, d^2\Omega^* \, d^3\hat{c}^* \tag{46}$$

and, thus, is not of the same form as (16) because of the presence of the unbalanced molecular Coriolis term

$$2\dot{Q}_{kl} Q_{ml} (c_m^* - v_m^*) \frac{\partial f^*}{\partial c_k^*} \tag{47}$$

Hence, in general, (46) may have solutions such that

$$f^* \neq f, \qquad \theta^* \neq \theta, \qquad \mathbf{v}^* \neq \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} \tag{48}$$

and

$$\mathbf{T}^* \neq \mathbf{Q}\mathbf{T}\mathbf{Q}^T, \qquad \mathbf{q}^* \neq \mathbf{Q}\mathbf{q}, \qquad \epsilon^* \neq \epsilon \tag{49}$$

It is thus clear that there is a fundamental difference between the kinetic theory of gases and continuum mechanics in so far as superimposed rigid rotations are concerned. Since the unbalanced molecular Coriolis term (47) has no analogous expression in continuum mechanics it does give reason to question the general validity of material frame indifference. However, to rigorously prove that material frame indifference is inconsistent with the kinetic theory of gases it is necessary to show that (46) has a solution where

$$\theta^* = \theta, \qquad \mathbf{v}^* = \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} \tag{50}$$

but

$$\mathbf{T}^* \neq \mathbf{Q}\mathbf{T}\mathbf{Q}^T, \qquad \mathbf{q}^* \neq \mathbf{Q}\mathbf{q} \tag{51}$$

Presently, it is not possible to resolve this question since no general solutions to the Boltzmann equation are known which give rise to solutions of the form (50) (i.e., solutions that differ by an *arbitrary time-dependent* rigid body rotation). Furthermore, if such solutions do exist they are probably rare because of the lack of completeness as alluded to in Truesdell.[12] Equations (46)–(49) merely demonstrate that the kinetic theory can give rise to inertial effects which have a different structure than those in continuum mechanics where material frame indifference is invoked. While this gives reason to question the general consistency of material frame indifference with the kinetic theory it does not represent conclusive evidence.

Although it is not presently possible to decisively resolve this issue concerning material frame indifference in the kinetic theory, it will now be demonstrated that this principle is in strong approximate agreement with the Boltzmann equation provided that the disparity in the time scales between the microscopic and macroscopic motions is extremely large—a situation which is usually the case in continuum mechanics. Equation (46) will be nondimensionalized as follows in order to accomplish this task:

$$f^+ = \frac{f^*}{f^*_{\max}}, \qquad \mathbf{c}^+ = \frac{\mathbf{c}^* t_0}{l_0}, \qquad \mathbf{x}^+ = \frac{\mathbf{x}^*}{l_0}$$
$$t^+ = \frac{t^*}{t_0}, \qquad \mathbf{u}^+ = \frac{\mathbf{u}^* t_0}{l_0} \tag{52}$$

where l_0 and t_0 are, respectively, the characteristic length and time scales associated with the microscopic motion (e.g., a mean free path and a mean free time). The body force per unit mass $\mathbf{b}^*$ and the rotation tensors $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ are external inputs which are specified independent of the molecular structure of the gas. Consequently, they should be nondimensionalized with respect to a *macroscopic* length scale L_0 and a macroscopic time scale T_0. Hence, we will introduce the dimensionless quantities

$$\mathbf{b}^+ = \frac{\mathbf{b}^* T_0^2}{L_0}, \qquad \dot{\mathbf{Q}}^+ = \dot{\mathbf{Q}} T_0, \qquad \mathbf{Q}^+ = \mathbf{Q} \tag{53}$$

By the judicious choice of t_0, l_0, T_0, and L_0, the dimensionless quantities in (52) and (53) can be made of order unity. The substitution of equations (52) and (53) into the Boltzmann equation (46) yields the dimensionless equation

$$\frac{\partial f^+}{\partial t^+} + c_k^+ \frac{\partial f^+}{\partial x_k^+} + \frac{t_0^2}{T_0^2}\frac{L_0}{l_0} b_k^+ \frac{\partial f^+}{\partial c_k^+}$$
$$+2\frac{t_0}{T_0}\dot{Q}_{kl}^+ Q_{ml}^+ u_m^+ \frac{\partial f^+}{\partial c_k^+} = \mathbb{C}^+[f^+] \tag{54}$$

where, in this case, the validity of the analysis is not dependent on the precise structure of the collision operator. Hence, if

$$\frac{t_0}{T_0} \ll 1 \tag{55}$$

then the unbalanced molecular Coriolis term on the left-hand side of (54) will have a negligible effect and results derived from the Boltzmann equation will be in at least strong approximate agreement with material frame indifference. For example, if we consider the case of hydrogen at standard temperature and pressure, the mean free time is 6.6×10^{-11} sec. We will examine the case where the gas is subjected to a steady rigid body rotation with centrifugal effects suppressed by the imposition of an external body force field. This type of example, in the presence of a radial temperature gradient, has been utilized by various authors to refute material frame indifference.[5,7] For such a case, the macroscopic time scale can be taken to be $1/\Omega$ where Ω is the angular velocity of the gas (this will guarantee that $\dot{\mathbf{Q}}^+$ is of order unity). Then, if we take the microscopic time scale to be a mean free time and restrict our attention to angular velocities $\Omega < 10^6$ rad/sec (an enormously rapid rotation), it is clear that

$$\frac{t_0}{T_0} < 10^{-6}$$

and the unbalanced Coriolis term would have a negligible effect. Material frame indifference would then be valid in a strong approximate sense.

3. CONCLUSION

It has been proven mathematically that results obtained from the Chapman–Enskog iterative procedure are not closed with respect to Euclidean transformations and, hence, if a given macroscopic process is a member of the Chapman–Enskog class, one which differs by an arbitrary superimposed rigid body motion will not be. Thus, results obtained from the Chapman–Enskog method cannot have a direct bearing on the principle of material frame indifference since this principle deals with Euclidean transformations and a Euclidean transformation can take one outside of the domain of validity of the method. In order to resolve this question, the effect of superimposed rigid body motions on solutions of the Boltzmann equation was examined and comparisons were made with continuum mechanics where such motions can have no effect on the stress tensor and heat flux vector when material frame indifference is invoked. It was found that with regard to superimposed translational accelerations of the gas, the kinetic theory is in support of the principle of material frame indifference and, thus, the invariance group of constitutive equations should be at least the extended Galilean group. However, there is reason to question the general consistency of the kinetic theory with material frame indifference when time-dependent rotations are included. This doubt arises from the presence of an unbalanced molecular Coriolis term which has no analog in continuum mechanics. Although it was not possible to resolve this question in full generality at this time because of the lack of exact solutions to the Boltzmann equation, it was proven that provided the disparity in the time scales between the microscopic and macroscopic motions is extremely large, the kinetic theory is in agreement with material frame indifference in at least a strong approximate sense.

The very existence of constitutive equations in the classical continuum mechanics sense where there is no dependence on initial or boundary conditions and the response of the material is local, requires that the disparity in the time and length scales associated with the microscopic and macroscopic motions be large. Hence, within the axiomatic framework of classical continuum mechanics (and for kinetic theory models with sufficiently small mean free times), it appears that the Boltzmann equation supports the imposition of material frame indifference as an axiom since they are consistent to within the same close approximation. In fact, when two disparate theories of nature such as the kinetic theory and continuum mechanics are applied to the same phenomenon, it is simply ridiculous to expect them to be in any more than strong approximate agreement. However, in recent years, continuum mechanics has been broadened to include the description of large scale nonlocal effects (cf., Edelen *et al.*[13] and Eringen[14]) where there can be an internal characteristic length (i.e., a microscale) which is much larger in comparison to the geometrical scale of the continuum than in the classical theories. Such theories could begin to approach the case of a highly rarefied gas where the microscale (e.g., the mean free path) can be of the order of the geometrical scale of the gas. The results of this paper indicate that for such a case there are reasons to question the validity of a continuum theory (if it could be applied) where material frame indifference is invoked. Future research will be required to resolve this issue with complete certainty.

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